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Matched asymptotics for two-dimensional planing hydrofoils with spoilers

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The purpose of the paper is to demonstrate the effectiveness of the matched asymptotic expansions (MAE) method for the planing flow problem. The matched asymptotics, taking into account the flow nonlinearities in those regions where they are most pronounced (i.e. in the vicinity of the edges), are shown to significantly extend the range where the linear theory gives good results. Two model problems are used: the planing flat plate with a spoiler on the trailing edge and the curved planing foil. Asymptotic solutions obtained by the MAE method are compared with those obtained using linear and exact nonlinear theories. Based on the results, the asymptotic solution to the planing problem under the gravity is proposed.

1. Introduction

The main difficulty of the free-surface flow problem (including the planing problem) is connected with nonlinearity of the boundary conditions which are, moreover, to be satisfied on surfaces of unknown geometry. Therefore we have to use assumptions simplifying the problem in question. If we neglect the gravity influence, the solution to the planing foil problem can be achieved through the methods of the theory of jets in an ideal fluid (Gurevich 1979). Such solutions are correct in the vicinity of the planing foil, but lose their correctness at large distance from the foil, where the free surface level tends to negative infinity logarithmically: $y \sim -\log|x|$ as $x \to \pm\infty$ (the Green paradox; in the case of the flat-plate hydrofoil, see Gurevich & Yampolsky 1933 and Green 1936). That is why the problem facing us has no uniqueness for it is impossible to set the immersion depth of the trailing edge above the undisturbed level of the free surface. It is to be underlined that the solution gets considerably more complicated in the case of a curved foil and at the same time of course, it remains non-unique (Gurevich 1979).

The formalism of the matched asymptotic expansions (MAE) method which uses an exact nonlinear solution to the two-dimensional flow problem without gravity as the inner asymptotic description is one of the effective ways to remove this drawback, for instance see Shen & Ogilvie (1972); Wu (1967); Ting & Keller (1974, 1977). The planing lifting-line solution to the three-dimensional flow problem without gravity is used as the outer expansion in Shen & Ogilvie (1972), i.e. the influence of the finite span is taken into consideration. The problem of small perturbations of free-surface flow under gravity caused by hydrodynamic singularities on its boundary is used as the outer expansion in Rispin (1967), Wu (1967) and Ting & Keller (1974), that is the influence of gravity at large distance from the foil is taken into consideration. As a

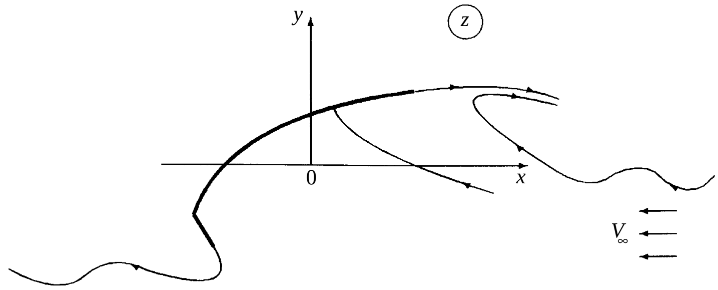

FIGURE 1. Planing foil with spoiler under gravity.

'zero' level for the free surface appeared at infinity, after the matching procedure the composite solutions to both problems become unique.

Another simplifying assumption to be introduced is the linearization procedure for the nonlinear boundary (dynamic and kinematic) conditions. The main postulate of the linear theory is that the planing hydrofoil brings small perturbations into the flow under gravity. The linearization procedure has to be accomplished both on the free surface and on the wetted portion of the foil. The linear planing problem has been considered by many authors. For instance see the works by Sedov (1979), Sretensky (1977), Khaskind (1943), Chaplygin (1940), Maruo (1951). It is well known that all those solutions were found in terms of linear theory adequately describing the flow pattern for small incidence angles and curvatures of the foils everywhere except the vicinity of the leading edge. The stagnation point and the sprinkle jet occur in this region and therefore the perturbations cannot be considered to be small there. The asymptotic analysis has shown that the linear solution loses its correctness in the proximity of the leading edge at distances of $O(\alpha^2)$, where α denotes the incidence angle. The linear solution is not correct in the vicinity of the trailing edge with the spoiler, either. The scale of the area of incorrectness is of $O(\bar{\varepsilon})$, where $\bar{\varepsilon}$ denotes the relative spoiler length (Rozhdestvensky & Fridman 1991).

The purpose of the present paper is to construct a composite asymptotic solution to the problem of the flow past a curved planing hydrofoil with a spoiler under gravity by using the MAE method (figure 1). The linear planing foil problem is considered to be the outer description and the spoiler and leading-edge flow problems are considered to be the inner ones. The composite solution is shown to be correct in the whole flow domain and allows us to remove some drawbacks of the linear theory (Chaplygin 1940; Maruo 1951; Sedov 1979). The so-called nonlinear edge correction technique used in the present research has proved its effectiveness for a wide range of free-surface flow problems: those for the cavitating hydrofoils with rounded noses (Furuya & Acosta 1973) and sharp ones (Plotkin 1978) and for cavitating wings (Achkinadze & Fridman 1994). The technique of leading-edge correction in the form of the multiplicative term was also applied by Kinnas (1991) for partially cavitating foils.

The spoiler, which is located on the trailing edge, on one hand brings the new difficulties into the solution to the problem, but on the other hand emphasizes the advantages of the proposed method. The problem of determining the influence of the spoiler upon the hydrodynamic coefficients of the planing surfaces has been considered by Fridman (1990) and Rozhdestvensky & Fridman (1990).

Finally some aspects of the two-dimensional non-steady planing problem were treated in Faltinsen (1996).

2. Statement of the problem

The planing hydrofoil of chord C and incidence angle α is assumed to be in the uniform flow of ideal incompressible fluid under gravity. V_∞ and p_0 denote the stream velocity and pressure at infinity. A Cartesian coordinate system (x, y) has its origin at the middle point of the foil's wetted length projection onto the stream direction which coincides with the x-axis direction, the y-axis being directed vertically upwards. The problem was non-dimensionalized by dividing all the length variables by half of the wetted length projection $a = l/2$ and velocity variables by V_∞. The Froude number based on the length l was introduced: $Fr = V_\infty/(gl)^{1/2}$, where g denotes the gravity acceleration. The spoiler which is mounted upon the trailing edge has the relative length $\bar{\varepsilon}$ and inclination angle $\beta_s \in [0; \pi)$.

The problem is considered to be solved when the velocity potential function $\varphi(x, y)$ is found. The harmonic function $\varphi(x, y)$, being the real part of an analytical function of complex potential $F = \varphi + \mathrm{i}\psi$, has to satisfy the boundary kinematic condition on the wetted length

$$\frac{\partial \varphi}{\partial n} = -\sin\theta(x),$$

the dynamic condition on the free surface

$$\frac{\partial \varphi}{\partial x} - \frac{1}{2}\left(\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2\right) - \nu y = 0,$$

and the condition at infinity, where $\partial/\partial n$ denotes the derivative in the normal direction, $\theta(x)$ denotes the tangential angle to the foil at point x, and $\nu = ga/V_\infty = 1/(2Fr^2)$.

3. Model problem – planing without gravity

Subdividing the flow domain into a far field (at distances of $O(1)$) and near fields (in the vicinity of the edges) we apply the MAE method.

3.1. *Linear planing theory – outer asymptotic description*

3.1.1. *Classical approach*

Assume that $Fr \to \infty$ (i.e. $\nu \to 0$) and that the values of $\bar{\varepsilon} = \varepsilon/a$ and $\theta(x)$ are small ($\bar{\varepsilon} = o(1)$, $\theta(x) = o(1)$, or $\bar{\varepsilon} \ll 1$, $\theta(x) \ll 1$). Under those circumstances we can neglect the second-order terms in the equations in §2 and can formulate the linearized planing flow problem without gravity as a Keldysh–Sedov boundary problem for the function of a complex conjugate velocity (the derivative of the complex velocity potential $F(z)$ with respect to the variable z)

$$\omega^{\mathrm{o}}(z) = u^{\mathrm{o}} - \mathrm{i}v^{\mathrm{o}} = \frac{\mathrm{d}F}{V_\infty \mathrm{d}z}$$

which is analytic in the upper semi-plane $z = x + \mathrm{i}y$ (i.e. in the domain occupied by the fluid) and is to satisfy the following conditions in the solution class $\infty - \infty$ (figure 2):

$$\begin{aligned} u^{\mathrm{o}} &= -1 \quad &&\text{for} \quad x \in]-\infty; -1[\cup]1; +\infty[, \\ v^{\mathrm{o}} &= -\theta(x) \quad &&\text{for} \quad x \in]-1; 1[. \end{aligned}$$

This is the statement of the outer problem. The class $\infty-\infty$ means that the function $\omega^{\mathrm{o}}(z)$ has to have square-root singularities for $z = \pm 1$. The function $\theta(x) = f'(x)$,

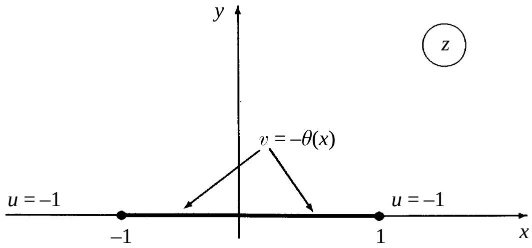

FIGURE 2. Outer linear planing problem.

where $y = f(x)$ denotes the foil geometry. The solution to this mixed-boundary problem can be carried out using the Keldysh–Sedov formulae:

$$\omega^{o}(z) = -1 + \frac{1}{\pi}\left(\frac{z+1}{z-1}\right)^{1/2} \int_{-1}^{1} \left(\frac{\zeta-1}{\zeta+1}\right)^{1/2} \frac{\theta(\zeta)}{\zeta - z} \mathrm{d}\zeta + \frac{B}{(1-z^2)^{1/2}} \tag{1}$$

where the value of the coefficient B is to be determined through the matching procedure. Division of the real and imaginary parts of (1) as ζ and z approach the real axis gives for $x \in [-1;1]$

$$u^{o}(x) = -1 + \frac{1}{\pi}\left(\frac{1+x}{1-x}\right)^{1/2} \int_{-1}^{1} \left(\frac{1-\xi}{1+\xi}\right)^{1/2} \frac{\theta(\xi)}{\xi - x} \mathrm{d}\xi + \frac{B}{\left(1-x^2\right)^{1/2}} \tag{2}$$

and therefore the pressure distribution coefficient is

$$C_p^{o} = \frac{2}{\pi}\left(\frac{1+x}{1-x}\right)^{1/2} \int_{-1}^{1} \left(\frac{1-\xi}{1+\xi}\right)^{1/2} \frac{\theta(\xi)}{\xi - x} \mathrm{d}\xi + \frac{2B}{\left(1-x^2\right)^{1/2}}. \tag{3}$$

In the case of planing flat-plate, $\theta(x) = \alpha$, the equations (1)–(3) give

$$\omega^{o}(z) = -1 - \mathrm{i}\alpha\left(1 - \left(\frac{z+1}{z-1}\right)^{1/2}\right) + \frac{B}{\left(1-z^2\right)^{1/2}}, \tag{4}$$

$$u^{o}(x) = -1 + \alpha\left(\frac{1+x}{1-x}\right)^{1/2} + \frac{B}{\left(1-x^2\right)^{1/2}}, \tag{5}$$

$$C_p^{o}(x) = 2\alpha\left(\frac{1+x}{1-x}\right)^{1/2} + \frac{2B}{\left(1-x^2\right)^{1/2}}. \tag{6}$$

It is obvious that the outer asymptotic description (1)–(6) is not correct as $z \to \pm 1$. This is the classical approach to the two-dimensional linear planing problem. The analytical or numerical solution to the problem involves, for the arbitrary function $\theta(x)$, some difficulties due to integration procedure.

3.1.2. *Non-quadrature approach*

The simple non-quadrature approach can be proposed for a wide range of the functions $\theta(x)$. This approach is based upon the method introduced by Terentev (1972) for supercavitating hydrofoils.

It is obvious that the new function

$$\Omega(z) = \omega^{o}(z) + 1 - \mathrm{i}\theta(z) \tag{7}$$

has pure real values on the wetted portion (as $x \in [-1;1]$, $y = 0$) and pure imaginary values on the free surfaces, see figure 2. Let us assume that the wetted portion of the planing foil is a polynomial

$$f(x) = \sum_{i=0}^{n} a_i x^i$$

and therefore

$$\theta(x) = f'(x) = \sum_{i=1}^{n} i a_i x^{i-1}$$

It should be pointed out that the multiplicity of a pole at infinity $z \to \infty$ for the function $\Omega(z)$ is equal to $(n-1)$, i.e. to the degree of the polynomial $f'(x)$. As the function $\omega^{\mathrm{o}}(z) \sim 1/\left(z^2-1\right)^{1/2}$ as $z \to \pm 1$, the similar behaviour of $\Omega(z)$ dictates the following solution to the mixed problem with homogeneous boundary conditions ($\infty - \infty$ class):

$$\Omega(z) = \mathrm{i}\left(\frac{z+1}{z-1}\right)^{1/2} \sum_{i=0}^{n-1} A_i z^i + \frac{B}{\left(1-z^2\right)^{1/2}} \tag{8}$$

where A_i ($i = 0, \ldots, n-1$) and B are the unknowns of the solution. Therefore one can obtain

$$\omega^{\mathrm{o}}(z) = -1 + \Omega(z) + \mathrm{i}\theta(z).$$

Unknown coefficients can be derived from the set of conditions

$$\lim_{z\to\infty} \frac{\omega^{\mathrm{o}}(z)+1}{z^i} = 0, \qquad i = 0, \ldots, n-1 \tag{9}$$

which results in

$$\begin{aligned} A_{n-1} &= -n a_n; \\ A_{n-2} &= -(n-1)a_{n-1} + n a_n; \\ A_{n-3} &= -(n-2)a_{n-2} + (n-1)a_{n-1} - \frac{n}{2}\, a_n; \\ &\vdots \end{aligned}$$

The value of B is to be determined from the matching procedure.

$\Omega(z)$ has a simple pole at infinity $z \to \infty$ in the case of a symmetric parabola $f(x) = 2h(1-x^2) + \alpha x$ as $x \in [-1;1]$ ($n = 2$, $a_2 = -2h$, $a_1 = \alpha$, $a_0 = 2h$), where α denotes the incidence angle. Finally, we arrive at the formulae

$$\omega^{\mathrm{o}}(z) = -1 - \mathrm{i}\left(\frac{z+1}{z-1}\right)^{1/2} (\alpha + 4h - 4hz) + \mathrm{i}\alpha - 4\mathrm{i}hz + \frac{B}{\left(1-z^2\right)^{1/2}}, \tag{10}$$

$$u^{\mathrm{o}}(x) = -1 + \left(\frac{1+x}{1-x}\right)^{1/2} (\alpha + 4h - 4hx) + \frac{B}{\left(1-x^2\right)^{1/2}}, \quad \text{as } x \in [-1;1]. \tag{11}$$

3.2. *Inner nonlinear descriptions*

3.2.1. *Spoiler problem*

It was mentioned above that outer asymptotic expansion loses its correctness in the vicinity of the spoiler as $z \to -1$. The picture of the flow shown in figure 3 corresponds to stretching of local coordinates in this region by a factor $1/\bar{\varepsilon}$: $X_1 = (x+1)/\bar{\varepsilon}$, $Y_1 = y/\bar{\varepsilon}$, $\bar{\varepsilon} \to 0$. The region occupied by the fluid is bounded by the rigid boundaries

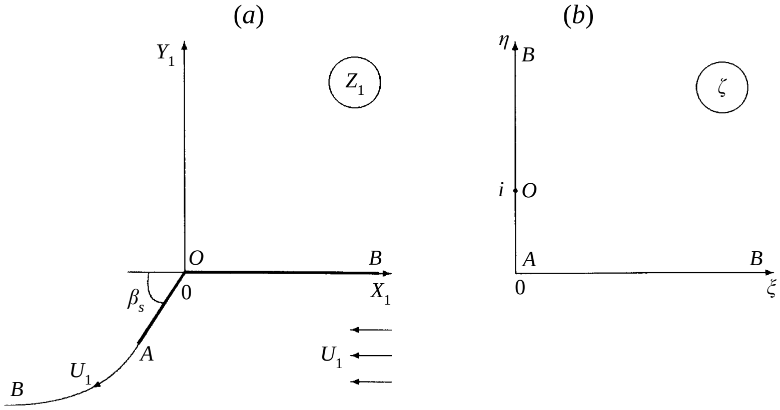

FIGURE 3. Flow pattern and auxiliary plane for the spoiler problem.

$[OB)$ and $[OA]$ and the free surface $\overset{\smile}{AB}$, the interval $[OA]$ being of the unit length $|OA| = 1$.

The problem arising is a two-dimensional nonlinear problem of the theory of jets in an ideal fluid. The solution to such a problem can be carried out by Chaplygin's method of singular points (Gurevich 1979) based on the idea of determining of a holomorphic function in complex plane, the function's zeros and poles being known and Liuville's theorem applied. Application of this method gives the following derivatives of the function of complex velocity potential $F_1 = \varphi_1 + i\psi_1$ in the auxiliary plane $\zeta = \xi + i\eta$:

$$\omega_1^i(\zeta) = \frac{dF_1}{U_1 dZ_1} = -\left(\frac{\zeta - i}{\zeta + i}\right)^{\beta_s/\pi}, \tag{12}$$

$$\frac{dF_1}{d\zeta} = N_1\zeta, \tag{13}$$

where N_1 denotes the real constant to be determined from

$$\frac{N_1}{U_1}\int_0^1 \eta\left(\frac{1+\eta}{1-\eta}\right)^{\beta_s/\pi} d\eta = \frac{N_1}{U_1}R = -1, \tag{14}$$

U_1 being the absolute value of the velocity on the free surfaces.

The pressure distribution coefficient ($0 \leqslant \eta < \infty$) is

$$C_{p1}^i = 1 - \left(\omega_1^i(\zeta)\right)^2 = 1 - \left|\frac{\eta - 1}{\eta + 1}\right|^{2\beta_s/\pi} \tag{15}$$

where the value of η is obtained from the relationships between physical and auxiliary coordinates on the interval $[OB]$, see figure 3(*a*):

$$X_1(\eta) = -\frac{N_1}{U_1}\int_1^\eta \eta\left(\frac{\eta+1}{\eta-1}\right)^{\beta_s/\pi} d\eta, \quad \eta \geqslant 1 \tag{16}$$

and on the interval $[OA]$:

$$|Z_1(\eta)| = -\frac{N_1}{U_1}\int_\eta^1 \eta\left(\frac{1+\eta}{1-\eta}\right)^{\beta_s/\pi} d\eta, \quad 0 \leqslant \eta \leqslant 1. \tag{17}$$

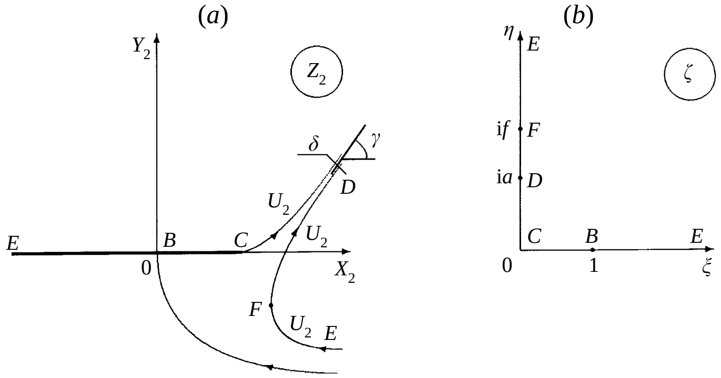

FIGURE 4. Flow pattern and auxiliary plane for the leading-edge problem.

The value of U_1 is the only unknown parameter in this problem to be determined through the matching procedure.

Note that Rozhdestvensky & Fridman (1991) contains a set of nonlinear solutions to the various models of the flow in the vicinity of the spoiler such as spoiler with a slot, with stagnation zone, etc.

3.2.2. *Leading-edge problem*

As outer description (1) is not correct in the proximity of the leading edge as $z \to 1$, stretched local coordinates were introduced in the form

$$X_2 = \frac{x-1}{\alpha^2} + L, \quad Y_2 = \frac{y}{\alpha^2}$$

where L is the distance of the stagnation point from the tip of the leading edge. The plane of the 'physical flow' and the auxiliary plane are depicted in figure 4. The region occupied by the fluid is bounded by the semi-infinite rigid boundary $[CE)$ and the free surfaces $\overset{\smile}{CD}$ and $\overset{\smile}{DFE}$. The incoming jet of semi-infinite width (the right-hand point E) is subdivided into two parts: one is beneath the rigid boundary (the 'left-hand' point E) and the second is the re-entrant jet (point D) with its image in the corresponding point $\zeta = \mathrm{i}a$. The point $\zeta = \mathrm{i}f$ is the image of the point F on the physical plane where the velocity vector is directed normally to the wetted line (i.e. vertically upward). Denote the width of the oncoming jet as δ, its angle at infinity as γ and the velocity on its boundary as U_2. Note also that the flow rate remains the same far up- and downstream, that is at the points E and D at infinity.

The Chaplygin method gives the derivatives of the function of the complex velocity potential $F_2 = \varphi_2 + \mathrm{i}\psi_2$ in the auxiliary plane $\zeta = \xi + \mathrm{i}\eta$:

$$\omega_2^i(\zeta) = \frac{\mathrm{d}F_2}{U_2 \mathrm{d}Z_2} = -\frac{\zeta - 1}{\zeta + 1}, \tag{18}$$

$$\frac{\mathrm{d}F_2}{\mathrm{d}\zeta} = N_2 \frac{\zeta(\zeta^2 - 1)}{\zeta^2 + a^2}. \tag{19}$$

Unknown constants N_2 and a are determined from the following conditions:

$$\omega_2^i(\mathrm{i}a) = \exp(-\mathrm{i}\gamma); \qquad \mathrm{i}q = \mathrm{i}U_2\delta = \oint_{\mathrm{i}a} \frac{\mathrm{d}F_2}{\mathrm{d}\zeta}\mathrm{d}\zeta$$

wherefrom

$$a = \tan\frac{\gamma}{2}; \qquad N_2 = \frac{2U_2\delta}{\pi(1+a^2)}. \tag{20}$$

Equations (18) and (19) can be combined to give

$$\begin{aligned} Z_2(\zeta) &= \int_1^{\zeta} \frac{dZ_2}{dF_2}\frac{dF_2}{d\zeta}d\zeta \\ &= -\frac{2\delta}{\pi(1+a^2)}\left(\frac{\zeta^2}{2} + 2\zeta + \frac{1-a^2}{2}\log(\zeta^2+a^2) - 2a\arctan\frac{\zeta}{a}\right)\Bigg|_1^{\zeta} \end{aligned} \tag{21}$$

and L is found in the form

$$L = \mathrm{Re}\,(Z_2(0)) = \frac{2\delta}{\pi}\cos^2\frac{\gamma}{2}\left(2.5 - (\pi-\gamma)\tan\frac{\gamma}{2} + \left(\tan^2\frac{\gamma}{2} - 1\right)\log\sin\frac{\gamma}{2}\right) \tag{22}$$

Thus, the three unknown parameters remain in the problem in question are δ, γ and U_2. All of them are to be determined through the matching procedure.

In the case of a horizontal oncoming jet $a \to 0$ and $f \to 1$ as $\gamma \to 0$ and the flow pattern coincides with that considered by Wagner (1932). The limiting form of (22) as $\gamma \to 0$ is

$$L^* = \mathrm{Re}\,(Z_2(\mathrm{i})) = \frac{6\delta}{\pi}. \tag{23}$$

The solution (18)–(19) gives another limiting case for the oncoming jet directed downstream ($a \to +\infty$ as $\gamma \to \pi$). It is obvious that $\delta \to \infty$ and one must change the stretching factor from $1/\alpha^2$ to $1/\alpha^4$. Then the flow pattern coincides with the one considered by Plotkin (1978) for the leading edge of a supercavitating plate, and equation (22) reduces for $\gamma \to \pi$

$$L = \tfrac{17}{48} \tag{24}$$

The pressure distribution coefficient on the wetted portion ($0 \leqslant \xi < +\infty$) for arbitrary γ is

$$C_{p2}^i = 1 - \left(\omega_2^i(\xi)\right)^2 = 1 - \left(\frac{\xi-1}{\xi+1}\right)^2 \tag{25}$$

where the value of ξ is obtained from the relationships between physical and auxiliary coordinates on the interval CE, see figure 4(a):

$$X_2(\zeta) = -\frac{2\delta}{\pi(1+a^2)}\left(-\frac{\xi^2}{2} - 2\xi + 2.5 + \frac{a^2-1}{2}\log\frac{\xi^2+a^2}{1+a^2} + 2a\left(\arctan a - \arctan\frac{a}{\xi}\right)\right). \tag{26}$$

3.3. *The matching procedure*

Generally speaking, the matching procedure is carried out in two steps: the first one allows us to match asymptotic descriptions in the outer region and in the trailing-edge region (the velocity U_1 in expression (7) and the coefficient B in equation (1) are determined) and the second one matches the outer description with a known value of B and leading-edge expansion to yield the values of δ, γ and U_2 in (18) and (19).

The limiting form of (2) as $x \to -1$ is

$$u^{oi}(x) \sim -1 + \frac{B}{(2(1+x))^{1/2}}.$$

On the other hand one can obtain from (16) that $X_1 \sim -N_1\eta^2/(2U_1)$ as $X_1 \to +\infty$ and $\eta \to +\infty$ and therefore the function $u_1^i = \mathrm{Re}\left(\omega_1^i(\zeta)\right)$ has the following limiting form as $X_1 \to +\infty$:

$$u_1^{io} \sim -1 + \frac{2\beta_s}{\pi}\frac{1}{\eta} \sim -1 + \frac{\sqrt{2}\,\beta_s}{\pi}\left(\frac{-N_1}{U_1X_1}\right)^{1/2}.$$

Taking into account the fact that $X_1 = (x+1)/\bar{\varepsilon}$, these two expressions are compared to give

$$U_1 = V_\infty, \quad B = \frac{2\beta_s}{\pi}\left(\frac{-N_1}{U_1}\bar{\varepsilon}\right)^{1/2} = \frac{2\beta_s}{\pi}\left(\frac{\bar{\varepsilon}}{R}\right)^{1/2}. \tag{27}$$

The limiting form of (2) as $x \to +1$ is

$$u^{oi}(x) \sim -1 + \left(2\int_{-1}^{1}\frac{\theta(\zeta)}{(1-\xi^2)^{1/2}}d\xi + B\right)\Big/\,(2(1-x))^{1/2}, \tag{28a}$$

the value of the coefficient B being governed by (27). The corresponding expression derived from the non-quadrature approach is

$$u^{oi}(x) \sim -1 + \left(B - 2\sum_{i=0}^{n-1}A_i\right)\Big/\,(2(1-x))^{1/2}. \tag{28b}$$

The asymptotic expansion of (26) gives

$$X_2 \sim -\frac{\delta}{\pi}\xi^2\cos^2\frac{\gamma}{2} \qquad \text{as} \quad X_2 \to -\infty \quad \text{and} \quad \xi \to +\infty$$

and therefore the behaviour of the function $u_2^i = \mathrm{Re}\left(\omega_2^i(\zeta)\right)$ is

$$u_2^{io} \sim -1 + \frac{2}{\xi} \sim -1 + 2\cos\frac{\gamma}{2}\left(\frac{\delta}{\pi}\right)^{1/2}\frac{1}{(-X_2)^{1/2}}. \tag{29}$$

The matching procedure for (28) and (29) considering the fact that $X_2 = (x-1)/\alpha^2 + L$ results in

$$U_2 = V_\infty, \quad \delta = \frac{\pi}{8\cos^2\gamma/2}\left(2\int_{-1}^{1}\frac{\theta(\zeta)}{\left(1-\xi^2\right)^{1/2}}\mathrm{d}\xi + B\right)^2\frac{1}{\alpha^2}. \tag{30a}$$

The matching procedure for (28) and (29) results in the similar expressions

$$U_2 = V_\infty, \quad \delta = \frac{\pi}{8\cos^2\gamma/2}\left(B - 2\sum_{i=0}^{n-1}A_i\right)^2\frac{1}{\alpha^2}. \tag{30b}$$

The value of L can be derived from the formula (22). One can see that the only parameter, namely the angle of the oncoming jet γ, remains unknown and that is why the solution to the problem under consideration defined everywhere is not unique. This fact was already mentioned to be an intrinsic feature of two-dimensional planing problems without gravity. In further consideration the angle $\gamma \in [0;\pi]$ is considered to be the parameter of solution. It is shown that the numerical value of γ does not significantly affect the hydrodynamic characteristics.

It should be noted that the distance of the stagnation point from the tip of the leading edge for the oncoming jet directed downstream (i.e. as $\gamma \to 0$) tends to infinity, but nevertheless the distance L^* of the stagnation point from the projection of the point F onto the X_2-axis (see figure 4a) could be used instead of L:

$$L^* = \frac{3}{4}\left(2\int_{-1}^{1}\frac{\theta(\zeta)}{\left(1-\xi^2\right)^{1/2}}\mathrm{d}\xi + B\right)^2 \frac{1}{\alpha^2} \tag{31a}$$

or, for the non-quadrature approach:

$$L^* = \frac{3}{4}\left(B - 2\sum_{i=0}^{n-1} A_i\right)^2 \frac{1}{\alpha^2}. \tag{31b}$$

Expressions (30a) and (31a) or correspondingly (30b) and (31b) reduce for the planing flat-plate at incidence angle $\theta(x) = \alpha$ as $\gamma \to 0$ to

$$\left.\begin{aligned} \delta &= \frac{\pi}{8}\left(2+\frac{B}{\alpha}\right)^2, \\ L^* &= \frac{6\delta}{\pi} = \frac{3}{4}\left(2+\frac{B}{\alpha}\right)^2 = 3\left(1+\frac{\beta_s}{\pi}\left(\frac{\bar{\varepsilon}}{R\alpha^2}\right)^{1/2}\right)^2. \end{aligned}\right\} \tag{32}$$

Note that the 'information stream' in the matching procedure is directed from the trailing edge to the leading one. In fact, the spoiler geometry parameters β_s and $\bar{\varepsilon}$ dictate the value of the coefficient B which, in turn, defines the sprinkle jet characteristics.

The additive composite solution for the pressure distribution coefficient can be constructed in the following manner:

$$C_p^c(x) = 1 - (u^c)^2 \tag{33}$$

where $u^c(x) = u^{\mathrm{o}} + u_1^i + u_2^i - u_1^{i\mathrm{o}} - u_2^{i\mathrm{o}}$. In the case of the flat plate with the spoiler ($\gamma = 0$) we arrive at the following expressions:

$$u^{\mathrm{o}}(x) = -1 + \alpha\left(\frac{1+x}{1-x}\right)^{1/2} + \frac{2\beta_s}{\pi}\left(\frac{\bar{\varepsilon}}{R}\right)^{1/2}\frac{1}{\left(1-x^2\right)^{1/2}},$$

$$u_1^i(x) = -\left(\frac{\eta-1}{\eta+1}\right)^{\beta_s/\pi}, \quad \text{where} \quad x = \frac{\bar{\varepsilon}}{R}\int_1^{\eta}\eta\left(\frac{\eta+1}{\eta-1}\right)^{\beta_s/\pi}\mathrm{d}\eta - 1,$$

$$x = 1 - \left(\alpha + \frac{\beta_s}{\pi}\left(\frac{\bar{\varepsilon}}{R}\right)^{1/2}\right)^2\left(\frac{2}{\left(1+u_2^i\right)^2} + \frac{2}{1+u_2^i} - 1 + \log\frac{1-u_2^i}{1+u_2^i}\right),$$

$$u_1^{\mathrm{o}i}(x) = -1 + \frac{\beta_s}{\pi}\left(\frac{2\bar{\varepsilon}}{R}\right)^{1/2}\frac{1}{(1+x)^{1/2}},$$

$$u_2^{\mathrm{o}i}(x) = -1 + \sqrt{2}\left(\alpha + \frac{\beta_s}{\pi}\left(\frac{\bar{\varepsilon}}{R}\right)^{1/2}\right)\frac{1}{(1-x)^{1/2}}.$$

3.4. *Exact solutions to nonlinear planing problems*

Two exact nonlinear solutions are presented in this subsection: the first is the solution to the problem of the planing flat plate with the spoiler and the second is the solution to the problem of the planing arched foil. Both of them are useful to verify asymptotic results. Note that the latter was obtained in close cooperation with S. M. Shebalov.

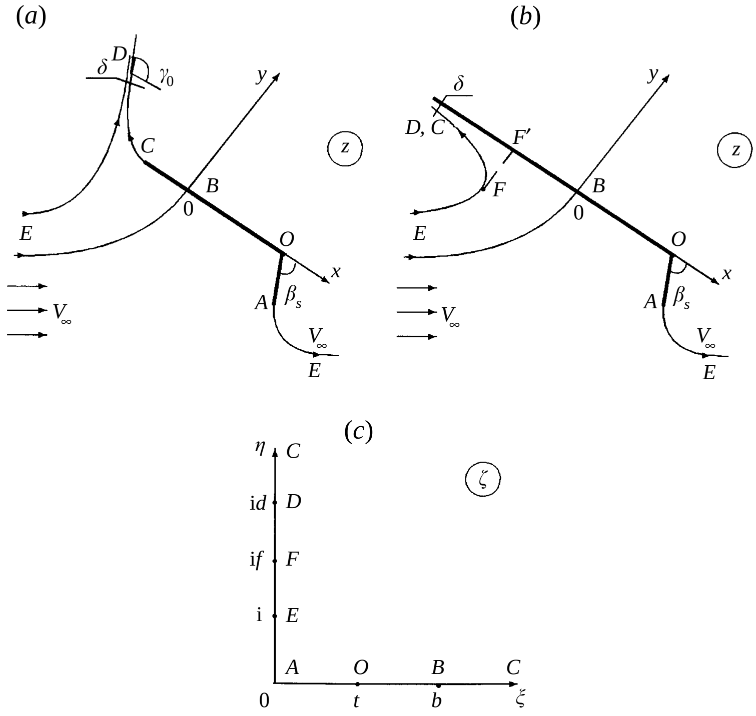

FIGURE 5. Flow pattern and auxiliary plane for the planing flatplate with the spoiler.

3.4.1. *Planing flat plate with spoiler*

Consider the nonlinear planing problem without gravity as $\theta(x) = \alpha$. Based on the classical work of Gurevich & Yampolsky (1933) we introduce a new Cartesian coordinate system which has its origin in the stagnation point, the x-axis coinciding with the flat plate, see figure 5. As before the spoiler on the trailing edge is of length ε and has an inclination angle β_s. The geometrical parameters of the sprinkle jet are δ and γ_0. The conjugate velocity $\omega^{\mathrm{n}}(z)$ is to satisfy the free streamline condition of constant speed and the condition at infinity $\omega^{\mathrm{n}} \to \exp(-\mathrm{i}\alpha)$ as $x \to \infty$.

With the correspondence between the physical $z = x + \mathrm{i}y$ plane and auxiliary quadrant $\zeta = \xi + \mathrm{i}\eta$ shown in figure 5, the Chaplygin method allows us to write down the following solution:

$$\omega^{\mathrm{n}}(\zeta) = \frac{\mathrm{d}F}{V_\infty \mathrm{d}z} = -\frac{\zeta - b}{\zeta + b}\left(\frac{\zeta - t}{\zeta + t}\right)^{\beta_s/\pi}, \tag{34}$$

$$\frac{\mathrm{d}F}{\mathrm{d}\zeta} = N\frac{\zeta(\zeta^2 - b^2)}{(\zeta^2 + d^2)(\zeta^2 + 1)^2}. \tag{35}$$

The equations (34) and (35) can be combined to give

$$z(\zeta) = \int_b^\zeta \frac{\mathrm{d}z}{\mathrm{d}F}\frac{\mathrm{d}F}{\mathrm{d}\zeta}\mathrm{d}\zeta = -\frac{N}{V_\infty}\int_b^\zeta \frac{\zeta(\zeta^2 + b^2)}{(\zeta^2 + d^2)(\zeta^2 + 1)^2}\left(\frac{\zeta + t}{\zeta - t}\right)^{\beta_s/\pi}\mathrm{d}\zeta. \tag{36}$$

Five conditions

$$l = \int_t^\infty \frac{\mathrm{d}z}{\mathrm{d}F}\frac{\mathrm{d}F}{\mathrm{d}\zeta}\mathrm{d}\zeta, \qquad \varepsilon\exp(-\mathrm{i}\beta_s) = \int_0^t \frac{\mathrm{d}z}{\mathrm{d}F}\frac{\mathrm{d}F}{\mathrm{d}\zeta}\mathrm{d}\zeta,$$

$$\mathrm{i}\delta V_\infty = \frac{1}{2}\oint_{id} \frac{\mathrm{d}F}{\mathrm{d}\zeta}\mathrm{d}\zeta, \qquad \omega^{\mathrm{n}}(\mathrm{i}) = \exp(-\mathrm{i}\alpha), \qquad \omega^{\mathrm{n}}(\mathrm{i}d) = \exp(-\mathrm{i}\gamma_0)$$

give the following system of five nonlinear transcendental equations in six unknowns b, d, δ, t, γ_0, N/V_∞:

$$l = -\frac{N}{V_\infty}\int_t^\infty \frac{\xi(\xi^2+b^2)}{(\xi^2+d^2)(\xi^2+1)^2}\left(\frac{\xi+t}{\xi-t}\right)^{\beta_s/\pi}\mathrm{d}\xi, \tag{37}$$

$$\varepsilon = -\frac{N}{V_\infty}\int_0^t \frac{\xi(\xi^2+b^2)}{(\xi^2+d^2)(\xi^2+1)^2}\left(\frac{\xi+t}{\xi-t}\right)^{\beta_s/\pi}\mathrm{d}\xi, \tag{38}$$

$$2\arctan\frac{b}{d} + \frac{2\beta_s}{\pi}\arctan\frac{t}{d} = \pi - \gamma_0, \tag{39}$$

$$2\arctan b + \frac{2\beta_s}{\pi}\arctan t = \pi - \alpha, \tag{40}$$

$$\frac{N}{V_\infty} = \frac{2\delta}{\pi}\frac{(d^2-1)^2}{(d^2+b^2)}. \tag{41}$$

It is easy to see that the solution is not unique for the number of unknowns is greater than the number of conditions. It should be mentioned that this problem can be interpreted as a supercavitating problem in the case of a given value of δ, the solution then being unique. The special case of the problem under consideration of $\gamma_0 = \pi$ and $\beta_s = \pi/2$, see figure 5(b), was analysed by Luckashevsky *et al.* (1978).

Equation (39) gives $d \to \infty$ as $\gamma_0 \to \pi$ and $\beta_s \in [0;\pi]$. As the 'physical' length of the flatplate $l \to \infty$ too, the wetted length l_w is introduced instead, the value of l_w being determined according to Wagner (1932):

$$l_w = |F'O| = \frac{2\delta}{\pi}\mathrm{Re}\left\{\int_t^{\mathrm{i}f} \frac{\xi(\xi^2+b^2)}{(\xi^2+d^2)(\xi^2+1)^2}\left(\frac{\xi+t}{\xi-t}\right)^{\beta_s/\pi}\mathrm{d}\xi\right\} \tag{42}$$

where the value of f can be derived from the condition $\omega^{\mathrm{n}}(\mathrm{i}f) = -\mathrm{i}$ written down in the form

$$\arctan\frac{b}{f} + \frac{\beta_s}{\pi}\arctan\frac{t}{f} = \frac{\pi}{4}. \tag{43}$$

The pressure distribution coefficient on the wetted portion is

$$C_p^{\mathrm{n}}(\xi) = 1 - \omega^{\mathrm{n}}(\xi)\overline{\omega^{\mathrm{n}}}(\xi) = 1 - \left(\frac{\xi-b}{\xi+b}\right)^2\left|\frac{\xi-t}{\xi+t}\right|^{2\beta_s/\pi} \tag{44}$$

where $\xi = \mathrm{Re}(\zeta)$ is connected to z through the relation (36).

The total force P acting on the planing flat plate with the spoiler is calculated by integration of $C_p^{\mathrm{n}}(\xi)$ as

$$P = -\frac{\mathrm{i}\rho V_\infty^2}{2}\int_0^\infty \left(1 - \omega^{\mathrm{n}}(\xi)\overline{\omega^{\mathrm{n}}}(\xi)\right)\frac{\mathrm{d}z}{\mathrm{d}\zeta}\mathrm{d}\zeta \tag{45}$$

This formula gives the expressions for the lift and drag coefficients

$$C_L = \frac{2\delta}{l_w}\left(b + \frac{\beta_s}{\pi}\frac{t(1+b^2)}{1+t^2} - \sin\alpha\right), \quad C_D = \frac{2\delta}{l_w}(1+\cos\alpha). \tag{46}$$

Let us analyse the solution to the problem under consideration in the case of

$$\gamma_0 \to \pi, \quad \alpha \to 0 \quad \text{and} \quad \frac{\varepsilon}{l} \to 0.$$

It is clear from what was done above that $b \to +\infty$ and $t \to 0^+$. Taking account of only linear terms in the equations (34)–(46) under those circumstances results in

$$t \sim \frac{1}{2}\left(\frac{\varepsilon}{a}\frac{1}{R}\right)^{1/2}, \quad R = \int_0^1 \xi \left(\frac{1+\xi}{1-\xi}\right)^{\beta_s/\pi}, \quad l_w = 2a, \tag{47}$$

$$\frac{|F'B|}{a} \sim \frac{12}{b^2} \sim 3\left(\alpha + \frac{2\beta_s}{\pi}t\right)^2, \tag{48}$$

$$\frac{\mathrm{d}F}{V_\infty \mathrm{d}z} \sim 1 - \frac{2}{b}\zeta - \frac{2\beta_s}{\pi}\frac{t}{\zeta}, \quad \frac{\mathrm{d}F}{\mathrm{d}\zeta} \sim -\frac{2\delta V_\infty b^2}{\pi}\frac{\zeta}{(\zeta^2+1)^2}, \tag{49}$$

$$C_L \sim \pi\alpha + 4\beta_s t. \tag{50}$$

The conformal mapping of the first quadrant of the ζ-plane onto the upper semi-plane u has the form

$$\zeta = \mathrm{i}\left(\frac{u+1}{u-1}\right)^{1/2}.$$

Substituting this expression into (49) gives

$$\frac{\mathrm{d}F}{V_\infty \mathrm{d}z} = 1 - \mathrm{i}\alpha\left(\frac{u+1}{u-1}\right)^{1/2} + \frac{4\beta_s}{\pi}\mathrm{i}t\frac{1}{\left(u^2-1\right)^{1/2}}. \tag{51}$$

Note that, with the difference between the coordinate systems, equation (51) is the same as equation (4), where the value of B is determined from (27). Moreover, the asymptotic and nonlinear approaches give similar formulae for the distance of the stagnation point from the tip of the leading edge (L^* and $|F'B|/a$ in (32) and (48) respectively).

3.4.2. *Planing arched foil*

The solution gets significantly more complicated when considering the nonlinear planing problem without gravity in the case of an arbitrary $\theta(x)$ for it is impossible to put the function $\theta(\zeta)$ on the part of the auxiliary plane ζ which corresponds to the rigid surface and therefore it is impossible to solve the boundary problem there.

Thus we have to determine not only the conformal transformation, which will transform the flow region z into the auxiliary plane ζ, but also the boundary conditions. Let us use the modification of the Sedov method by Gurevich (1979) to solve the planing problem depicted in the figure 6(a). The plane of the complex potential $F = \varphi + \mathrm{i}\psi$ and the auxiliary upper half-plane ζ are shown in figure 6(b) and figure 6(c) respectively.

With the correspondence between the planes, the derivative of the complex potential in the ζ-plane is

$$\frac{\mathrm{d}F}{\mathrm{d}\zeta} = N\frac{\zeta}{\zeta - d} \tag{52}$$

where $\zeta = d$ is the image of the point D. The value of N is determined from the

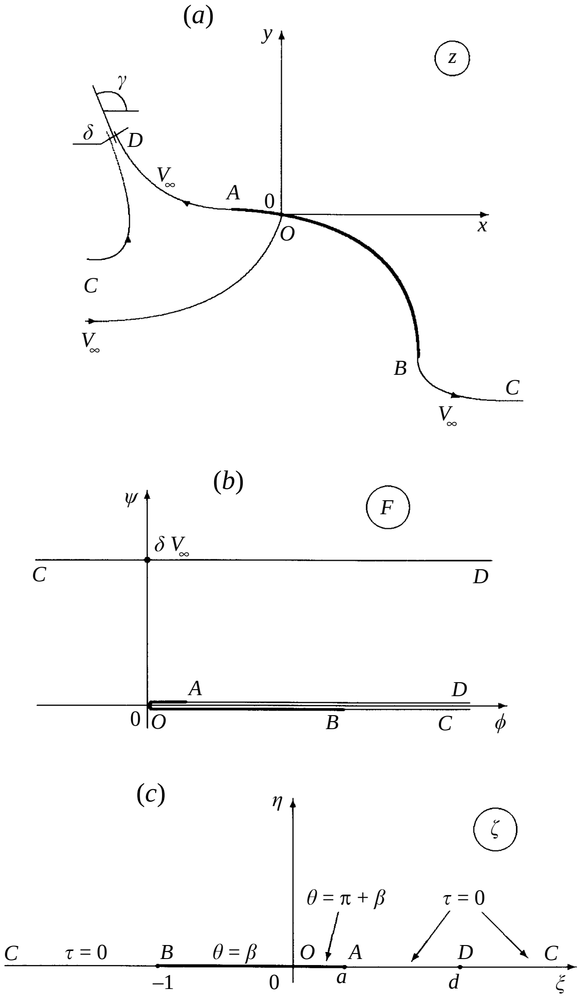

FIGURE 6. (*a*) Flow pattern for the planing arched foil. (*b*) The plane of the complex potential. (*c*) The corresponding auxiliary plane.

following condition of mass conversation (q_D denotes the strength of the sprinkle jet):

$$\mathrm{i}q_D = \mathrm{i}\delta V_\infty = -\frac{1}{2}\oint_d \frac{\mathrm{d}F}{\mathrm{d}\zeta}\mathrm{d}\zeta$$

from which

$$N = -\frac{\delta V_\infty}{\pi d}. \tag{53}$$

The complex conjugate velocity $\mathrm{d}F/\mathrm{d}z$ can be written down in the form

$$\frac{\mathrm{d}F}{\mathrm{d}z} = V\exp(-\mathrm{i}\Theta) = V_\infty \exp(-\mathrm{i}\lambda) \tag{54}$$

where V is the velocity absolute value and λ denotes the Levi–Civita function

$$\lambda = \Theta + \mathrm{i}\log\frac{V}{V_\infty} = \Theta + \mathrm{i}\tau. \tag{55}$$

The function λ is analytic in the upper half-plane ζ and tends to zero at infinity. The mixed boundary problem arising for the function $\lambda(\zeta)$ is shown in figure 6(*c*). The function $\theta(x)$ is connected to the foil topography $y = f(x)$ through the relation $\theta = \arctan(\mathrm{d}f/\mathrm{d}x)$.

The Keldysh–Sedov formula gives the following solution to this problem in the 0–0 class:

$$\lambda(\zeta) = -\frac{1}{\pi}\,((\zeta+1)(\zeta-a))^{1/2}\left\{\int_{-1}^{a}\frac{\theta(t)}{[(1+t)(a-t)]^{1/2}}\frac{\mathrm{d}t}{t-\zeta}+\int_{0}^{a}\frac{\pi}{[(1+t)(a-t)]^{1/2}}\frac{\mathrm{d}t}{t-\zeta}\right\} \tag{56}$$

which exists if and only if

$$\frac{1}{\pi}\int_{-1}^{a}\frac{\theta(t)}{[(1+t)(a-t)]^{1/2}}\mathrm{d}t+\int_{0}^{a}\frac{\mathrm{d}t}{[(1+t)(a-t)]^{1/2}}=0. \tag{57}$$

Note that the function $\theta(\xi)$ is an unknown one and the integrals in equations (56) and (57) cannot be calculated yet. The parameters of the conformal transformation $\zeta(z)$ (i.e. the variables a and d) are to be determined from the conditions

$$\left.\frac{\mathrm{d}F}{\mathrm{d}z}\right|_{\zeta=d}=V_\infty\exp(-\mathrm{i}\gamma),\quad \left.\frac{\mathrm{d}F}{\mathrm{d}z}\right|_{\zeta=\infty}=V_\infty \tag{58}$$

or, which is the same

$$\mathrm{Re}\{\lambda(d)\}=\gamma,\quad \mathrm{Re}\{\lambda(\infty)\}=0.$$

Using equation (56) those conditions can be rewritten to give

$$-\frac{1}{\pi}\,[(d+1)(d-a)]^{1/2}\int_{-1}^{a}\frac{\theta(t)}{[(1+t)(a-t)]^{1/2}\,(t-d)}\mathrm{d}t+\frac{\pi}{2}+\arcsin\frac{d(a-1)+2a}{d(a+1)}=\gamma. \tag{59}$$

$$\frac{1}{\pi}\int_{-1}^{a}\frac{\theta(t)}{[(1+t)(a-t)]^{1/2}}\mathrm{d}t+\frac{\pi}{2}-\arcsin\frac{1-a}{1+a}=0. \tag{60}$$

It should be pointed out that the last expression is the same as the equation (57).

Introducing the arc coordinate $s(\xi)$ for the wetted portion of the planing foil so that $0 \leqslant s \leqslant S$ (where S denotes the whole wetted length and the value of s increases from point A to point B, figure 6) gives the relationship

$$\theta=\arctan\left(\frac{\mathrm{d}y/\mathrm{d}s}{\mathrm{d}x/\mathrm{d}s}\right).$$

This equation can be used in conjunction with the equality

$$\frac{\mathrm{d}z}{\mathrm{d}s}=\exp(\mathrm{i}\theta)$$

to give

$$s(\xi)=-\frac{\delta}{d\pi}\int_{a}^{\xi}\exp\{-\mathrm{i}\theta(t)\}\exp\{\mathrm{i}\lambda(t)\}\frac{t\mathrm{d}t}{\zeta-d}. \tag{61}$$

Substituting the solution (56) as $1 \leqslant \xi \leqslant a$ into (61) finally gives

$$s(\xi)=-\frac{\delta}{d\pi}\int_{\xi}^{a}G(t,a,d)\mathrm{d}t, \tag{62}$$

where

$$G(t,a,d) = \frac{2\,[a(1+t)(a-t)]^{1/2} + t(a-1) + 2a}{(1+a)(t-d)} \times \exp\left\{ \frac{[(1+t)(a-t)]^{1/2}}{\pi} \oint_{-1}^{a} \frac{\theta(t)}{[(1+t)(a-t)]^{1/2}} \frac{\mathrm{d}t}{t-\xi} \right\}. \tag{63}$$

The function $s(\xi)$ clearly has to satisfy the condition

$$s(-1) - S = 0. \tag{64}$$

Thus there are four unknown parameters a, d, δ and γ and the unknown function $\theta(\xi)$ $(1 \leqslant \xi \leqslant a)$ in the problem under consideration. On the other hand there are four conditions to obtain the solution to the problem: (59), (60), (64) and the singular integral equation (62). Just as in the previous subsection, it is easy to see that the solution is not unique for the number of unknowns is greater than the number of conditions. Nevertheless, the numerical analysis has shown that the problem can be solved for the given value of γ which does not significantly affect the hydrodynamic coefficients.

4. Planing under gravity – asymptotic approach

Basing upon the asymptotic method and results obtained in the previous section, let us consider the more general problem of the planing foil with the spoiler under gravity.

Following the MAE method, the flow domain is subdivided into far-field and near-field regions in the same manner as for the planing problem without gravity: at distances of $O(1)$ and in the vicinity of the edges.

4.1. *Outer problem – linear theory*

In the case of $(x,y) = O(1)$, $\alpha = o(1)$ and $\bar{\varepsilon} = o(1)$ one can use the method developed by Sedov (1979). It was shown Rozhdestvensky & Fridman (1990) that the presence of the spoiler on the trailing edge dictates the class of solution to the boundary problem, namely $\infty - \infty$. The function of the complex potential is

$$F(z) = V_\infty(F_1(z) - z) \tag{65}$$

where

$$F_1(z) = \mathrm{e}^{-\mathrm{i}\nu z}\left(\int_{+\infty}^{z}\left(\xi - \sum_{n=1}^{\infty} n a_n (t - (t^2-1)^{1/2})^n\right)\frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\nu t}}{(t^2-1)^{1/2}}\mathrm{d}t + A\right) \tag{66}$$

and $A = A_1 + \mathrm{i}A_2$ denotes the wave amplitude at infinity as $x \to +\infty$, parameter ν being $\nu = ga/V_\infty = 1/(2Fr^2)$ as in §2.

The value of ξ is derived from the Loran-series expansion for the following composition

$$\frac{\mathrm{d}F_1}{\mathrm{d}z} + \mathrm{i}\nu F_1 = \frac{\mathrm{i}\xi}{z} + \frac{\mathrm{i}\xi_1}{z^2} + \cdots. \tag{67}$$

The values of the real coefficients a_n, $(n = 1, 2, \ldots)$ are to be determined from the algebraic system of linear equations:

$$\left.\begin{aligned}
&(2n+1)a_{2n+1} + b_{2n+1} + \frac{2v}{\pi}\sum_{i=0}^{\infty}\left(\frac{1}{4(\mathrm{i}+n+1)^2-1} - \frac{1}{4(\mathrm{i}-n)^2-1}\right)a_{2i+1} = 0,\\
&2na_{2n} + b_{2n} + \frac{2v}{\pi}\sum_{i=1}^{\infty}\left(\frac{1}{4(\mathrm{i}+n)^2-1} - \frac{1}{4(\mathrm{i}-n)^2-1}\right)a_{2i} = 0,
\end{aligned}\right\} \tag{68}$$

where b_n $(n = 1, 2, \ldots)$ are the coefficients of the Fourier-series expansion:

$$-\theta(x) - v^2\int_1^x\int_1^{x_1}\theta(x_2)\mathrm{d}x_2\mathrm{d}x_1 + v^2\zeta(1-x) - v\eta - v\xi\sigma = \frac{1}{\sin\sigma}\sum_{n=1}^{\infty} b_n \sin n\sigma$$

and $\sigma = \arccos x$, $\theta(x)$ denotes the tangential angle to the foil at point x.

The following notation was used in the previous formula:

$$\left.\begin{aligned}
&\eta = \mathrm{Re}(F_1(1)) = A_1\cos v + A_2 \sin v - \xi Q_0 + \sum_{n=1}^{\infty} na_n Q_n(v),\\
&\zeta = \mathrm{Im}(F_1(1)) = -A_1 \sin v + A_2\cos v + \xi P_0 - \sum_{n=1}^{\infty} na_n P_n(v),\\
&P_n(v) + \mathrm{i}Q_n(v) = \mathrm{e}^{-\mathrm{i}v}\int_{+\infty}^{1}\frac{\left(t - \left(t^2-1\right)^{1/2}\right)^n}{\left(t^2-1\right)^{1/2}}\mathrm{e}^{\mathrm{i}vt}\mathrm{d}t.
\end{aligned}\right\} \tag{69}$$

Note that $a_n = O(n^{-3})$ as $n \to \infty$. The coefficients a_n can be interpreted as holomorphic functions of η, ζ and ξ for given $v \geqslant 0$.

The conjugate velocity $\omega^{\mathrm{o}}(z) = u^{\mathrm{o}} - \mathrm{i}v^{\mathrm{o}}$ is

$$\omega^{\mathrm{o}}(z) = \frac{\mathrm{d}F}{V_\infty \mathrm{d}z} = -1 - \mathrm{i}vF_1(z) + \frac{\mathrm{i}}{\left(z^2-1\right)^{1/2}}\left(\xi - \sum_{n=1}^{\infty} na_n\left(z - \left(z^2-1\right)^{1/2}\right)^n\right) \tag{70}$$

and on the wetted portion of the foil where $x \in [-1; 1]$, $y = 0$

$$u^{\mathrm{o}}(x) = -1 + \left(\sum_{n=1}^{\infty} na_n T_n(x) - \xi\right)\frac{1}{\left(1-x^2\right)^{1/2}}, \tag{71}$$

$$C_p^{\mathrm{o}} = \left(\sum_{n=1}^{\infty} na_n T_n(x) - \xi\right)\frac{2}{\left(1-x^2\right)^{1/2}}, \tag{72}$$

where $T_n(x) = \cos\arccos nx$ denote the Chebyshev polynomials.

To close the solution, Sedov introduced a condition at the trailing edge which is equivalent to the Kutta–Joukowsky one. In the case of $\bar{\varepsilon} > 0$ the Kutta–Joukowsky condition is substituted by the one provided by the matching procedure.

4.2. *Inner problems*

The outer expansion for the conjugate velocity $\omega^{\mathrm{o}}(z)$, derived in the framework of the linear theory, loses its correctness near the edges as $z \to \pm 1$, namely in the regions where $z - 1 = O(\alpha^2)$ and $z + 1 = O(\bar{\varepsilon})$. Similar results were obtained for the planing foil without gravity, $Fr = \infty$. There is just one, but significant, difference – one must take into account the influence of gravity in the inner problems. Nevertheless it is easy to see that after the stretching of the local coordinates in the vicinity of the leading and trailing edges by the factors $s_1 = 1/\alpha^2$ and $s_2 = 1/\bar{\varepsilon}$ respectively, the characteristic lengths in both regions are of $o(1)$. That is why the local Froude numbers are of the order of $Fr_1^i = O(s_1^{1/2})$ and $Fr_2^i = O(s_2^{1/2})$ (i.e. $Fr_1^i \gg 1$ and $Fr_2^i \gg 1$) and we can neglect gravity in the inner expansions. Thus the asymptotic solutions to the local problems without gravity (12)–(13) and (18)–(19) are used as inner expansions.

4.3. *The matching procedure*

The matching procedure is carried out in just the same way as in the case of planing without gravity. At the first stage the outer asymptotic expansion and the inner one in the spoiler vicinity are matched, the new condition being determined instead of the Kutta–Joukowsky one. The second stage allows the sprinkle jet parameters δ, γ and U_2 in the local problem in the proximity of the leading edge to be derived.

As the limiting form of (71) as $x \to -1$ is

$$u^{oi}(x) \sim -1 + \left(\sum_{n=1}^{\infty}(-1)^n n a_n - \xi\right)\frac{1}{[2(1+x)]^{1/2}},$$

the matching procedure results in

$$U_1 = V_\infty, \quad B = \sum_{n=1}^{\infty}(-1)^n n a_n - \xi = \frac{2\beta_s}{\pi}\left(\frac{\bar{\varepsilon}}{R}\right)^{1/2}. \tag{73}$$

On the other hand, the limiting form of (71) as $x \to +1$ is

$$u^{oi}(x) \sim -1 + \left(\sum_{n=1}^{\infty} n a_n - \xi\right)\frac{1}{[2(1-x)]^{1/2}},$$

and therefore

$$U_2 = V_\infty, \quad \delta = \frac{\pi}{8\cos^2\gamma/2}\left(\sum_{n=1}^{\infty} n a_n - \xi\right)\frac{1}{\alpha^2}. \tag{74}$$

The second equation in (73) gives the condition to close the outer linear problem. The application of this condition results in the following expression for the sprinkle jet thickness δ:

$$\delta = \frac{\pi}{8\cos^2\gamma/2}\left(\sum_{n=1}^{\infty}(2n-1)\frac{a_{2n-1}}{\alpha} + \frac{2\beta_s}{\pi}\left(\frac{\bar{\varepsilon}}{R}\right)^{1/2}\frac{1}{\alpha}\right)^2. \tag{75}$$

The distance of the stagnation point from the tip of the leading edge in the case of $\gamma = 0$ is

$$L = \frac{6\delta}{\pi} = \frac{3}{4}\left(\sum_{n=1}^{\infty}(2n-1)\frac{a_{2n-1}}{\alpha} + \frac{2\beta_s}{\pi}\left(\frac{\bar{\varepsilon}}{R}\right)^{1/2}\frac{1}{\alpha}\right)^2. \tag{76}$$

The composite additive expansion for the pressure distribution coefficient is constructed in the same manner as for the planing problem without gravity

$$C_p^c(x) = 1 - (u^c)^2 \tag{77}$$

where $u^c(x) = u^o + u_1^i + u_2^i - u_1^{io} - u_2^{io}$.

It should be pointed out that $a_1 = \alpha$ and $a_k = 0$, $k \geqslant 2$ as $Fr \to \infty$ and formulae (73)–(76) give the solution to the planing problem without gravity as a special case.

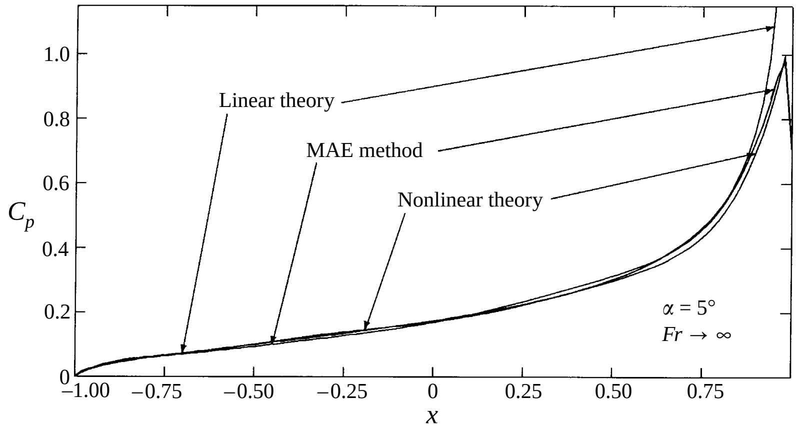

FIGURE 7. The pressure distribution coefficient C_p computed by means of LT, NLT and MAE, versus x for the planing flat plate.

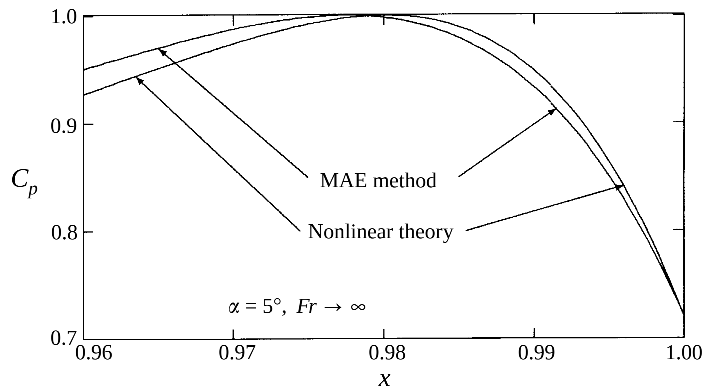

FIGURE 8. The pressure distribution coefficient C_p close to the leading edge.

5. Numerical results and discussion

The main idea is to compare the hydrodynamics coefficients computed within the framework of the linear theory (LT), the nonlinear one (NLT) and the asymptotic approach (AA). Two planing surfaces have been chosen: the planing flat plate with the spoiler and the planing arc of a parabola with the topography of the wetted portions $f(x) = \alpha x$ and $f(x) = 2h(1 - x^2) + \alpha x$ respectively.

It is well known that the linear theory of the planing surfaces gives satisfactory results only in the case of small values of the foil curvature and incidence angle. On the other hand this theory has provided powerful methods to solve effectively the problems in question. Nonlinear theory requires more complicated techniques even for the simple special cases. As shown, the asymptotic approach based on the MAE method combines the advantages of both linear and nonlinear theories: on one hand the asymptotic solution is of the same simplicity as the linear one and on the other hand the MAE method takes into account the nonlinear effects that considerably affect the hydrodynamic coefficients.

Most of the next set of figures contain the comparison between numerical results computed by mqeans of LT, NLT and AA. Figures 7 and 8 show the pressure distribution coefficient C_p versus x for the planing flat–plate without the spoiler, the incidence

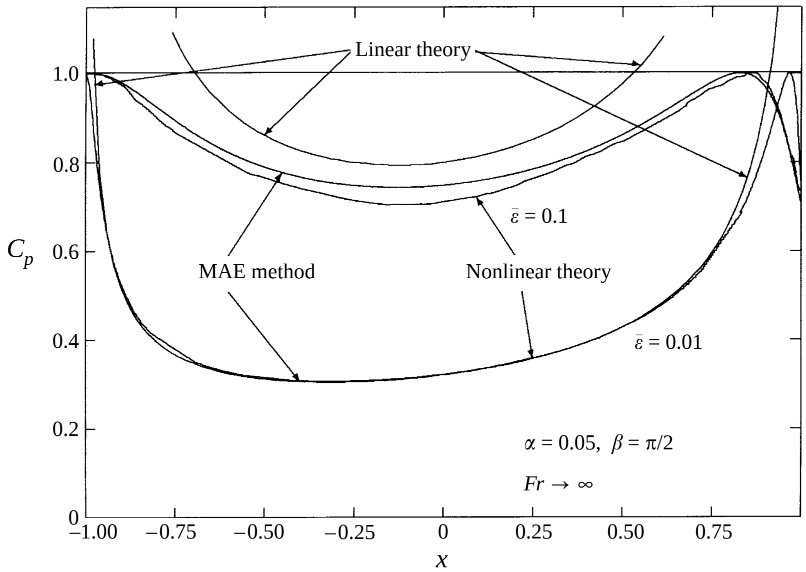

FIGURE 9. The pressure distribution coefficient C_p computed by means of LT, NLT and MAE, versus x for the planing flat plate with the spoiler.

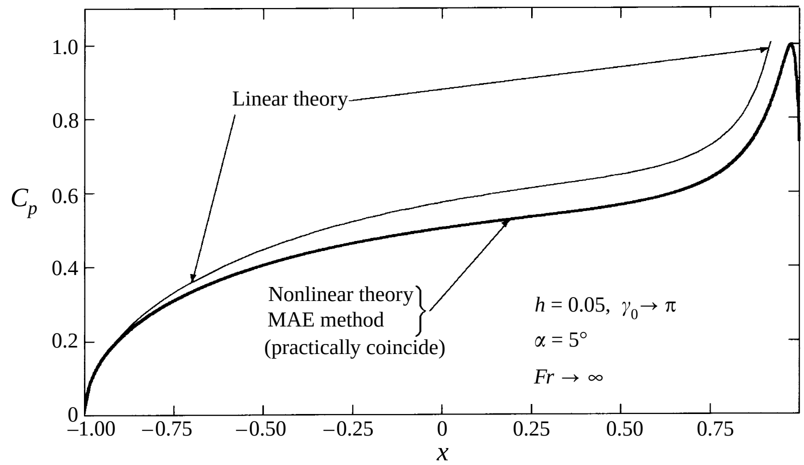

FIGURE 10. The pressure distribution coefficient C_p computed by means of LT, NLT and MAE, versus x for the planing parabola $f(x) = 2h(1 - x^2) + \alpha x$.

angle α being 5°. As the geometrical parameters are small, the linear theory is accurate everywhere except in the vicinity of the leading edge. The asymptotic solution approaches the nonlinear one even in the region where the linear theory gives a result coming arbitrarily close to infinity (figure 8). Comparing the pressure distribution coefficient $C_p(x)$ in the case of the flat plate with the spoiler for a wide range of incidence angle and spoiler length $\bar{\varepsilon}$ shows that the asymptotic and nonlinear solutions are nearly the same in the spoiler proximity too, the linear solution being unlimited in the vicinity of both edges $x \to \pm 1$. Corresponding numerical results are depicted in the figure 9.

Some calculations for the planing parabola $f(x) = 2h(1 - x^2) + \alpha x$ are presented in figures 10 and 11. The MAE method gives results compatible with nonlinear theory

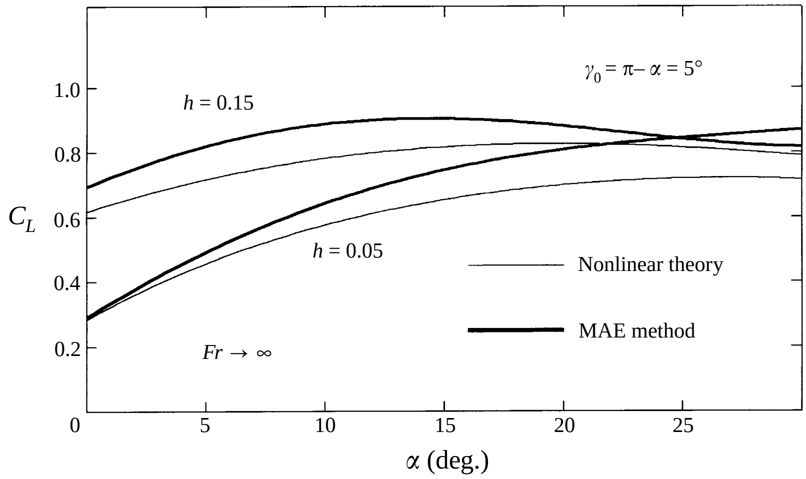

FIGURE 11. The lift coefficients C_L computed by means of NLT and MAE, versus incidence angle α for the planing parabola $f(x) = 2h(1 - x^2) + \alpha x$.

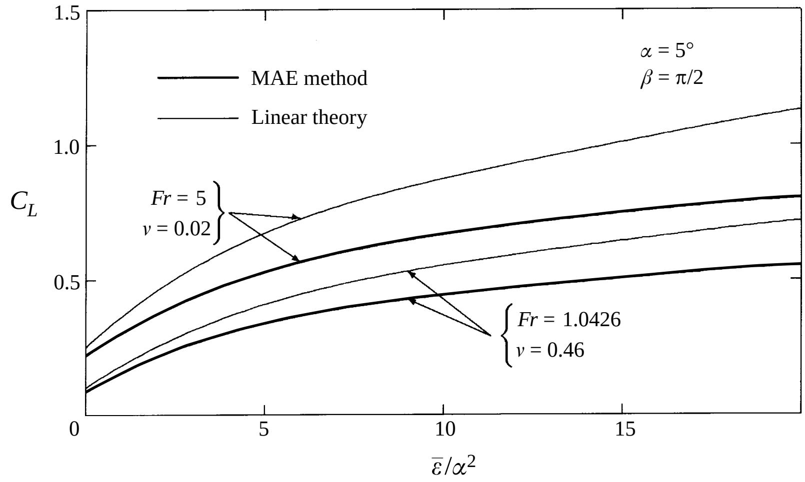

FIGURE 12. The lift coefficient C_L computed by means of LT and MAE for the planing flat plate with the spoiler under gravity.

for the pressure distribution and lift coefficients. It is assumed that the gravity force is not acting ($Fr \to \infty$) for the calculations presented in figures 7–11.

The calculations for the planing problem under gravity demonstrate the same: the asymptotic method shows much more satisfactory results (approaching the nonlinear results) than the linear theory, the level of complication of the solution procedure being the same in both cases. The lift coefficients C_L and $C_a = C_L/(\pi\nu)$ versus the spoiler parameter $\bar{\varepsilon}/\alpha^2$ are shown in figures 12 and 13 for the planing flat plate with the spoiler for Froude numbers $Fr = 0.40825$, 1.0426 and 5. The incidence angle equals 5° and the inclination angle for the spoiler equals to 90°. It should be mentioned that the lift coefficient C_a was introduced by Chaplygin (1940) to analyse the lifting capacity of the planing foil for small Froude numbers. Using five terms a_n, $n = 1, \ldots, 5$ in expression (72) seemed to be accurate enough for the numerical calculations.

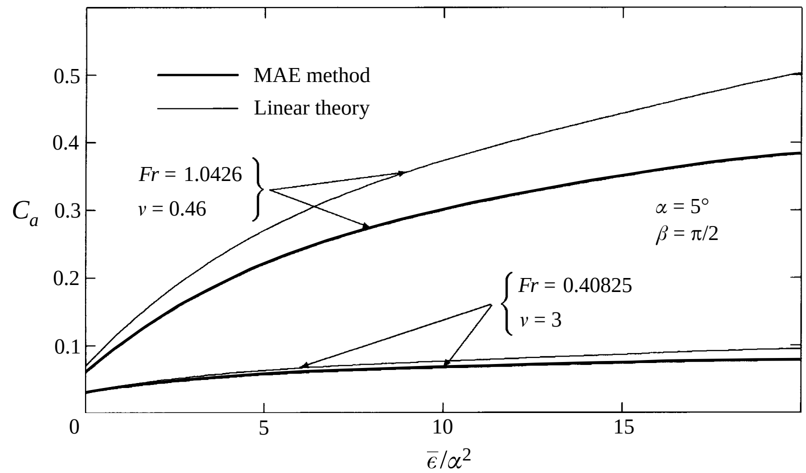

FIGURE 13. The lift coefficient C_a computed by means of LT and MAE for the planing flat plate with the spoiler under gravity.

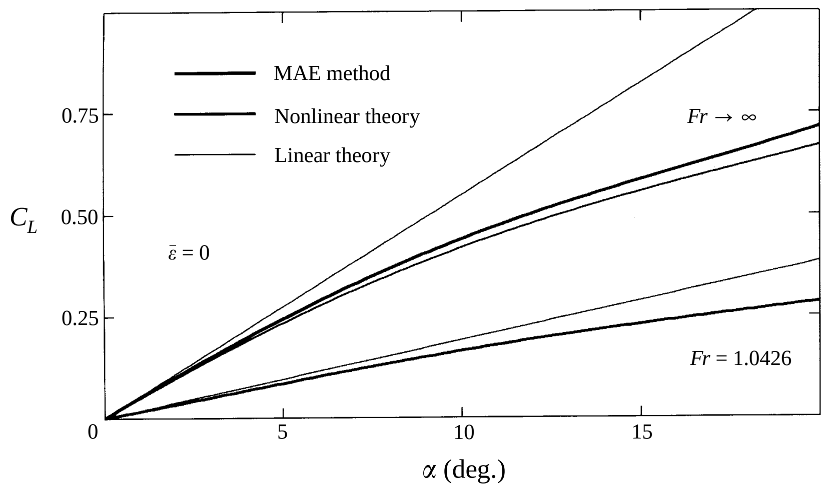

FIGURE 14. The comparison of the lift coefficients C_L versus the incidence angle α computed for the planing flat plate by means of LT, NLT and AA.

Figure 14 shows the comparison of the lift coefficients C_L versus the incidence angle α computed for the planing flat–plate by means of LT, NLT and AA. The lift calculated using NLT and AA had similar values in a wide range of angle α.

6. Conclusion

An asymptotic approach based on the matched asymptotic expansions method has been used to solve the two-dimensional planing flow problem under gravity. The linear planing theory is used for the outer description and the nonlinear theory of jets in an ideal fluid, applied to the problem arising near the edges, is used for the inner one. The matching procedure gives a composite solution to the problem under consideration that is valid everywhere. The detailed analysis of the planing problem without gravity is presented, a new non-quadrature linear approach being proposed.

Two exact nonlinear solutions are derived and are compared with the asymptotic solutions to the following problems: the planing flat plate with a spoiler and the planing arched foil. The numerical results show the effectiveness of the asymptotic approach proposed.

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